

LIPSCHITZ NORMAL EMBEDDINGS AND DETERMINANTAL SINGULARITIES

HELGE MØLLER PEDERSEN AND MARIA A. S. RUAS

ABSTRACT. The germ of an algebraic variety is naturally equipped with two different metrics up to bilipschitz equivalence. The inner metric and the outer metric. One calls a germ of a variety Lipschitz normally embedded if the two metrics are bilipschitz equivalent. In this article we prove that the model determinantal singularity, that is the space of $m \times n$ matrices of rank less than a given number, is Lipschitz normally embedded. We will also discuss some of the difficulties extending this result to the case of general determinantal singularities.

1. INTRODUCTION

If $(X, 0)$ is the germ of an algebraic (analytic) variety, then one can define two natural metrics on it. Both are defined by choosing an embedding of $(X, 0)$ into $(\mathbb{C}^N, 0)$. The first is the *outer metric*, where the distance between two points $x, y \in X$ is given by $d_{out}(x, y) := \|x - y\|_{\mathbb{C}^N}$, so just the restriction of the Euclidean metric to $(X, 0)$. The other is the *inner metric*, where the distance is defined as $d_{in}(x, y) := \inf_{\gamma} \{length_{\mathbb{C}^N}(\gamma) \mid \gamma: [0, 1] \rightarrow X \text{ is a rectifiable curve, } \gamma(0) = x \text{ and } \gamma(1) = y\}$. Both of these metrics are independent of the choice of the embedding up to bilipschitz equivalence. The outer metric determines the inner metric, and it is clear that $d_{out}(x, y) \leq d_{in}(x, y)$. The other direction is in general not true, and we say that $(X, 0)$ is *Lipschitz normally embedded* if the inner and outer metric are bilipschitz equivalent. *Bilipschitz geometry* is the study of the bilipschitz equivalence classes of these two metrics. Now one can of course define the inner and outer metric for any embedded subspace of Euclidean space, but the bilipschitz class might in this case depend of the embeddings.

In January 2016 Asuf Shachar asked the following question on Mathoverflow.org: Is the Lie group $GL_n^+(\mathbb{R})$ Lipschitz normally embedded, where $GL_n^+(\mathbb{R})$ is the group of $n \times n$ matrices with positive determinants. A positive answer was given by Katz, Katz, Kerner, Liokumovich and Solomon in [KKKLS16]. They first prove it for the *model determinantal singularity* $M_{n,n}^n$ (they call it the determinantal singularity), that is the set of $n \times n$ matrices with determinant equal to zero. Then they replace the segments of the straight line between two points of $GL_n^+(\mathbb{R})$ that passes through $GL_n^-(\mathbb{R})$ with a line arbitrarily close to $M_{n,n}^n$. Their proof relies on topological arguments, and some results on conical stratifications of MacPherson and Procesi [MP98]. In this article we give an alternative proof relying only on linear algebra and simple trigonometry, which also works for all model determinantal singularities. We will also discuss the case of general determinantal singularities, by giving some examples of determinantal singularities that are not Lipschitz normally embedded, and then discussing some additional assumptions on a determinantal singularity that might imply it is Lipschitz normal embedded.

2000 *Mathematics Subject Classification.* 14B05, 32S05, 32S25; 57M99.

Key words and phrases. Lipschitz geometry, Determinantal singularities.

This work is in the intersection of two areas that have seen a lot of interest lately, namely bilipschitz geometry and determinantal singularities. The study of bilipschitz geometry of complex spaces started with Pham and Teissier that studied the case of curves in [PT69]. It then lay dormant for long time until Birbrair and Fernandes began studying the case of complex surfaces [BF08]. Among important recent results are the complete classification of the inner metric of surfaces by Birbrair, Neumann and Pichon [BNP14], the proof that Zariski equisingularity is equivalent to bilipschitz triviality in the case of surfaces by Neumann and Pichon [NP14] and the proof that outer Lipschitz regularity implies smoothness by Birbrair, Fernandes, Lê and Sampaio [BFLS16]. Determinantal singularity is also an area that has been around for along time, that recently saw a lot of interest. They can be seen as a generalization of ICIS, and the recent results have mainly been in the study of invariants coming from their deformation theory. In [GZÈ09] Èbeling and Gusein-Zade defined the index of a 1-form, and the Milnor number have been defined in various different ways by Ruas and da Silva Pereira [SRDSP14], Damon and Pike [DP14] and Nuño-Ballesteros, Oréface-Okamoto and Tomazalla [NBOOT13]. Their deformation theory have also been studied by Gaffney and Rangachev [GR15] and Frühbis-Krüger and Zach [FZ15].

This article is organized as follows. In section 2 we discuss the basic notions of Lipschitz normal embeddings and determinantal singularities and give some results concerning when a space is Lipschitz normally embedded. In section 3 we prove the main theorem, that model determinantal singularities are Lipschitz normally embedded. Finally in section 4 we discuss some of the difficulties to extend this result to the settings of general determinantal singularities.

2. PRELIMINARIES ON BILIPSCHITZ GEOMETRY AND DETERMINANTAL SINGULARITIES

Lipschitz normal embeddings. In this section we discuss some properties of Lipschitz normal embeddings. We will first give the definition of Lipschitz normally embedding we will work with.

Definition 2.1. We say that X is *Lipschitz normally embedded* if there exist $K > 1$ such that for all $x, y \in X$,

$$(1) \quad d_{in}(x, y) \leq K d_{out}(x, y).$$

We call a K that satisfies the inequality a *bilipschitz constant* of X .

A trivial example of a Lipschitz normally embedded set is \mathbb{C}^n . For an example of a space that is not Lipschitz normally embedded, consider the plane curve given by $x^3 - y^2 = 0$, then $d_{out}((t^2, t^3), (t^2, -t^3)) = 2|t|^{\frac{3}{2}}$ but the $d_{in}((t^2, t^3), (t^2, -t^3)) = 2|t| + o(t)$, this implies that $\frac{d_{in}((t^2, t^3), (t^2, -t^3))}{d_{out}((t^2, t^3), (t^2, -t^3))}$ is unbounded as t goes to 0, hence there cannot exist a K satisfying (1).

Pham and Teissier [PT69] show that in general the outer geometry of a complex plane curve is equivalent to its embedded topological type, and the inner geometry is equivalent to the abstract topological type. Hence a plane curve is Lipschitz normally embedded if and only if it is a union of smooth curves intersecting transversely. See also Fernandes [Fer03].

In the cases of higher dimension the question of which singularities are Lipschitz normally embedded becomes much more complicated. It is no longer only rather trivial singularities that are Lipschitz normally embedded, for example in the case of surfaces the first author together with Neumann and Pichon, shows that rational surface singularities are Lipschitz normally embedded if and only if they are minimal

[NPP15]. As we will later see, determinantal singularities give examples of non trivial Lipschitz normally embedded singularities in arbitrary dimensions.

We will next give a couple of results about when spaces constructed from Lipschitz normally embedded spaces are themselves Lipschitz normally embedded. First is the case of product spaces.

Proposition 2.2. *Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ and let $Z = X \times Y \subset \mathbb{R}^{n+m}$. Z is Lipschitz normally embedded if and only if X and Y are Lipschitz normally embedded.*

Proof. First we prove the “if” direction. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. We need to show that $d_{in}^{X \times Y}((x_1, y_1)(x_2, y_2)) \leq K d_{out}^{X \times Y}((x_1, y_1)(x_2, y_2))$. Let K_X be the constant such that $d_{in}^X(a, b) \leq K_X d_{out}^X(a, b)$ for all $a, b \in X$, and let K_Y be the constant such that $d_{in}^Y(a, b) \leq K_Y d_{out}^Y(a, b)$ for all $a, b \in Y$. We get, using the triangle inequality, that

$$d_{in}^{X \times Y}((x_1, y_1)(x_2, y_2)) \leq d_{in}^{X \times Y}((x_1, y_1)(x_1, y_2)) + d_{in}^{X \times Y}((x_1, y_2)(x_2, y_2)).$$

Now the points (x_1, y_1) and (x_1, y_2) both lie in the slice $\{x_1\} \times Y$ and hence $d_{in}^{X \times Y}((x_1, y_1)(x_1, y_2)) \leq d_{in}^Y(y_1, y_2)$ and likewise we have $d_{in}^{X \times Y}((x_1, y_2)(x_2, y_2)) \leq d_{in}^X(x_1, x_2)$. This then implies that

$$d_{in}^{X \times Y}((x_1, y_1)(x_2, y_2)) \leq K_Y d_{out}^Y(y_1, y_2) + K_X d_{out}^X(x_1, x_2),$$

where we use that X and Y are Lipschitz normally embedded. Now it is clear that $d_{out}^{X \times Y}((x_1, y_1)(x_1, y_2)) = d_{out}^Y(y_1, y_2)$ and $d_{out}^{X \times Y}((x_1, y_2)(x_2, y_2)) = d_{out}^X(x_1, x_2)$. Also, since $d_{out}^{X \times Y}((x_1, y_1)(x_2, y_2))^2 = d_{out}^Y(y_1, y_2)^2 + d_{out}^X(x_1, x_2)^2$ by definition of the product metric, we have that $d_{out}^{X \times Y}((x_1, y_1)(x_1, y_2)) \leq d_{out}^{X \times Y}((x_1, y_1)(x_2, y_2))$ and $d_{out}^{X \times Y}((x_1, y_2)(x_2, y_2)) \leq d_{out}^{X \times Y}((x_1, y_1)(x_2, y_2))$. It then follows that

$$d_{in}^{X \times Y}((x_1, y_1)(x_2, y_2)) \leq (K_Y + K_X) d_{out}^{X \times Y}((x_1, y_1)(x_2, y_2)).$$

For the other direction, let $p, q \in X$ consider any path $\gamma: [0, 1] \rightarrow Z$ such that $\gamma(0) = (p, 0)$ and $\gamma(1) = (q, 0)$. Now $\gamma(t) = (\gamma_X(t), \gamma_Y(t))$ where $\gamma_X: [0, 1] \rightarrow X$ and $\gamma_Y: [0, 1] \rightarrow Y$ are paths and $\gamma_X(0) = p$ and $\gamma_X(1) = q$. Now $l(\gamma) \geq l(\gamma_X)$, hence $d_{in}^X(p, q) \leq d_{in}^Z((p, 0), (q, 0))$. Now Z is Lipschitz normally embedded, so there exist a $K > 1$ such that $d_{in}^Z(z_1, z_2) \leq K d_{out}^Z(z_1, z_2)$ for all $z_1, z_2 \in Z$. We also have that $d_{out}^Z((p, 0), (q, 0)) = d_{out}^X(p, q)$, since X is embedded in Z as $X \times \{0\}$. Hence $d_{in}^X(p, q) \leq K d_{out}^X(p, q)$. The argument for Y being Lipschitz normally embedded is the same exchanging X with Y . \square

Another case we will need later is the case of cones.

Proposition 2.3. *Let X be the cone over M , then X is Lipschitz normally embedded if and only if M is Lipschitz normally embedded.*

Proof. We first prove that M Lipschitz normally embedded implies that X is Lipschitz normally embedded.

Let $x, y \in X$ and assume that $\|x\| \geq \|y\|$. First if $x = 0$ (the cone point), then the straight line from y to x lies in X , hence $d_{in}(x, y) = d_{out}(x, y)$. So we can assume that $x \neq 0$, and let $y' = \frac{y}{\|y\|} \|x\|$. Then y' and x lie in the same copy M_ε of M , hence $d_{in}(x, y') \leq K_M d_{out}(x, y')$. Now y' is the point closest to y on M_ε . Hence all of $M_\varepsilon - y'$ lies on the other side of the affine hyperspace through y' orthogonal to the line $\overline{yy'}$ from y to y' . Hence the angle between $\overline{yy'}$ and the line $\overline{y'x}$ between y' and x is more than $\frac{\pi}{2}$. Therefore, the Euclidean distance from y to x is larger than $l(\overline{yy'})$ and $l(\overline{y'x})$. This gives us:

$$\begin{aligned} d_{in}(x, y) &\leq d_{in}(x, y') + d_{in}(y', y) \leq K_M d_{out}(x, y') + d_{out}(y', y) \\ &\leq (K_M + 1) d_{out}(x, y). \end{aligned}$$

For the other direction, assume that X is Lipschitz normally embedded, but M is not Lipschitz normally embedded.

Since M is compact the only obstructions to being Lipschitz normally embedded are local. So let $p \in M$ be a point such that M is not Lipschitz normally embedded in a small open neighbourhood $U \subset M$ of p . By Proposition 2.2 we have that $U \times (-\varepsilon, \varepsilon)$ is not Lipschitz normally embedded, where $0 < \varepsilon$ is much smaller than the distance from M to the origin. Now the quotient map from $c: M \times [0, \infty) \rightarrow X$ induces an outer (and therefore also inner) bilipschitz equivalence of $U \times (-\varepsilon, \varepsilon)$ with $c(U \times (-\varepsilon, \varepsilon))$. Since both U and ε can be chosen to be arbitrarily small, we have that there does not exist any small open neighbourhood of $p \in X$ that is Lipschitz normally embedded, contradicting that X is Lipschitz normally embedded. Hence X being Lipschitz normally embedded implies that M is Lipschitz normally embedded. \square

Determinantal singularities. Let $M_{m,n}$ be the space of $m \times n$ matrices with complex (or real) entries. For $1 \leq t \leq \min\{m, n\}$ let $M_{m,n}^t$ denote the *model determinantal singularity*, that is $M_{m,n}^t = \{A \in M_{m,n} \mid \text{rank } A < t\}$. $M_{m,n}^t$ is an algebraic variety, with algebraic structure defined by the vanishing of all $t \times t$ minors. It is homogeneous, and hence a real cone over its real link, it is also a complex cone but it is the real conical structure we will use. It is highly singular with the singular set of $M_{m,n}^t$ being $M_{m,n}^{t-1}$. In fact the action of the group $\text{GL}_m \times \text{GL}_n$ by conjugation insures that the decomposition $M_{m,n}^t = \bigcup_{i=1}^t M_{m,n}^i - M_{m,n}^{i-1}$ is a Whitney stratification.

Let $F: \mathbb{C}^N \rightarrow M_{m,n}$ be a map with holomorphic entries, then $X = F^{-1}(M_{m,n}^t)$ is a *determinantal variety* of type (m, n, t) if $\text{codim } X = \text{codim } M_{m,n}^t = (m - t + 1)(n - t + 1)$. If $F(0) = 0$ we will call the germ $(X, 0)$ a *determinantal singularity* of type (m, n, t) .

Determinantal singularities can have quite bad singularities, hence one often restrict to the following subset with better properties:

Definition 2.4. Let X be a determinantal singularity defined by a map $F: \mathbb{C}^N \rightarrow M_{m,n}$. One says that X is an *essentially isolated determinantal singularity* (EIDS for short) if F is transversal to the strata of $M_{m,n}^t$ at all point in a punctured neighbourhood of the origin.

While an EIDS can still have very bad singularities at the origin, its singular points away from the origin only depends on the type and N , for example $X = F^{-1}(M_{m,n}^1)$ has a nice action, and is stratified by $X^i = F^{-1}(M_{m,n}^i - M_{m,n}^{i-1})$. A lot of interesting singularities are EIDS, for example all ICIS are EIDS.

In proving that our singularities are Lipschitz normally embedded, we often have to change coordinates, to get some nice matrices for our points. Hence we need the following lemma to see that these changes of coordinates do preserve the inequalities we are using.

Lemma 2.5. *Let $V \subset M_{m,n}$ be a subset invariant under linear change of coordinates. If $x, y \in V$ satisfy $d_{\text{in}}(x, y) \leq K d_{\text{out}}(x, y)$, then the same is true after any linear change of coordinates.*

Proof. Any linear change of coordinates of $M_{m,n}$ is given by conjugation by a pair of matrices $(A, B) \in \text{GL}_m \times \text{GL}_n$.

First we see that the outer metric is just scale by the following:

$$\begin{aligned} d_{\text{out}}(Ax B^{-1}, Ay B^{-1}) &= \|A(x - y)B^{-1}\| = \|A\| \|x - y\| \|B^{-1}\| \\ &= \|A\| \|B^{-1}\| d_{\text{out}}(x, y). \end{aligned}$$

Second we consider the case of length of curves. Let $P_{x,y}^V := \{\gamma: [0,1] \rightarrow V \mid \gamma \text{ is a rectifiable curve such that } \gamma(0) = x \text{ and } \gamma(1) = y\}$, then the conjugation $\gamma \rightarrow A\gamma B^{-1}$ defines a bijection of $P_{x,y}^V$ and $P_{Ax B^{-1}, Ay B^{-1}}^{AV B^{-1}} = P_{Ax B^{-1}, Ay B^{-1}}^V$. Moreover, that $l(A\gamma B^{-1}) = \|A\|l(\gamma)\|B^{-1}\|$ follows from the definition of length of a curve. Hence

$$\begin{aligned} d_{in}(x, y) &= \inf_{\gamma \in P_{x,y}^V} \{l(\gamma)\} = \inf_{A\gamma B^{-1} \in P_{Ax B^{-1}, Ay B^{-1}}^{AV B^{-1}}} \left\{ \frac{l(A\gamma B^{-1})}{\|A\|\|B^{-1}\|} \right\} \\ &= \frac{d_{in}(Ax B^{-1}, Ay B^{-1})}{\|A\|\|B^{-1}\|}. \end{aligned}$$

The result then follows. \square

3. THE CASE OF THE MODEL DETERMINANTAL SINGULARITIES

In this section we prove that $M_{m,n}^t$ is Lipschitz normally embedded. We do that by considering several cases for the position of two points $p, q \in M_{m,n}^t$, and finding inequalities of the form $d_{in}(p, q) \leq K d_{out}(p, q)$, where we explicitly give the value of K . First we consider the simple case where $q = 0$.

Lemma 3.1. *Let $p \in M_{m,n}^t$ then $d_{in}(p, 0) = d_{out}(p, 0)$.*

Proof. This follows since $M_{m,n}^t$ is conical, and hence the straight line from p to 0 lies in $M_{m,n}^t$. \square

The second case we consider is when p and q are orthogonal. This case is not much more complicated than the case $q = 0$.

Lemma 3.2. *Let $p, q \in M_{m,n}^t$ such that $\langle p, q \rangle = 0$. Then $d_{in}(p, q) \leq 2 d_{out}(p, q)$.*

Proof. That $\langle p, q \rangle = 0$ implies that the line from p to q , the line from p to 0 and the line from q to 0 form a right triangle with the line from p to q as the hypotenuse. Hence $d_{out}(p, 0) \leq d_{out}(p, q)$ and $d_{out}(q, 0) \leq d_{out}(p, q)$, this gives that:

$$d_{in}(p, q) \leq d_{in}(p, 0) + d_{in}(q, 0) = d_{out}(p, 0) + d_{out}(q, 0) \leq 2 d_{out}(p, q).$$

\square

The last case we need to consider is the case where p and q are not orthogonal. This case is a little more complicated and we need to do the proof by induction.

Lemma 3.3. *Let $p, q \in M_{m,n}^t$ such that $\langle p, q \rangle \neq 0$. Then $d_{in}(p, q) \leq 2 \text{rank}(p) d_{out}(p, q)$.*

Proof. The proof is by induction in t by considering $M_{m,n}^t$ as depending on t as $M_{m'+t, n'+t}^t$. The base case is $M_{m,n}^1 = \{0\}$, which trivially satisfies the inequality. So we assume the theorem is true for $M_{m-1, n-1}^{t-1}$.

By a change of coordinates we can assume that p and q have the following forms:

$$p = \left(\begin{array}{c|ccc} p_{11} & p_{12} & \dots & p_{1n} \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & \end{array} \middle| \begin{array}{c} D_p \end{array} \right) \text{ and } q = \left(\begin{array}{c|ccc} q_{11} & 0 & \dots & 0 \\ \hline q_{21} & & & \\ \vdots & & & \\ q_{m1} & & & \end{array} \middle| \begin{array}{c} D_q \end{array} \right)$$

where $p_{11}, q_{11} \neq 0$ and $D_p, D_q \in M_{m-1, n-1}$. Then let p', q' and q_0 be the following points:

$$p' = \left(\begin{array}{c|ccc} q_{11} & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & \end{array} \middle| \begin{array}{c} D_p \end{array} \right), \quad q' = \left(\begin{array}{c|ccc} q_{11} & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & \end{array} \middle| \begin{array}{c} D_q \end{array} \right) \text{ and } q_0 = \left(\begin{array}{c|ccc} q_{11} & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & \end{array} \middle| \begin{array}{c} 0 \end{array} \right).$$

It is clear that $p', q' \in M_{m,n}^t$, moreover the straight line $\overline{pp'}$ from p to p' is in $M_{m,n}^t$, and the straight line $\overline{qq'}$ from q to q' is in $M_{m,n}^t$. Hence $d_{in}(p, p') = d_{out}(p, p')$ and $d_{in}(q, q') = d_{out}(q, q')$. Let H_{q_0} be the affine space through q_0 defined as

$$H_{q_0} := \left\{ \left(\begin{array}{c|ccc} q_{11} & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & \end{array} \middle| \begin{array}{c} A \end{array} \right) \in M_{m,n} \mid \text{where } A \in M_{m-1,n-1} \right\}.$$

It is clear that $p', q' \in H_{q_0}$ and hence $p', q' \in H_{q_0} \cap M_{m,n}^t$. Now $H_{q_0} \cap M_{m,n}^t$ is isomorphic to $M_{m-1,n-1}^{t-1}$, and we get by induction $d_{in}(p', q') \leq 2 \text{rank } D_p d_{out}(p', q') = 2(\text{rank } p - 1) d_{out}(p', q')$.

We now have that:

$$(2) \quad d_{in}(p, q) \leq d_{in}(p, q') + d_{in}(q', q) \leq d_{in}(p, p') + d_{in}(p', q') + d_{in}(q', q) \\ \leq d_{out}(p, p') + 2(\text{rank } p - 1) d_{out}(p', q') + d_{out}(q', q).$$

The line $\overline{pp'}$ is in the direction $p - p'$ and the line $\overline{p'q'}$ is in the direction $q' - p'$. These direction are

$$p - p' = \left(\begin{array}{c|ccc} p_{11} - q_{11} & p_{12} & \dots & p_{1n} \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & \end{array} \middle| \begin{array}{c} 0 \end{array} \right) \text{ and } q' - p' = \left(\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & \end{array} \middle| \begin{array}{c} D_q - D_p \end{array} \right),$$

hence $\overline{pp'}$ and $\overline{p'q'}$ are orthogonal. This implies that the straight line $\overline{pq'}$ is the hypotenuse of a right triangle given by p, p' and q' . We therefore have that $d_{out}(p, p') \leq d_{out}(p, q')$ and $d_{out}(p', q') \leq d_{out}(p, q')$.

Likewise we have that the line $\overline{pq'}$ is in the direction $p - q'$ and the line $\overline{qq'}$ is in the direction $q - q'$. These direction are

$$p - q' = \left(\begin{array}{c|ccc} p_{11} - q_{11} & p_{12} & \dots & p_{1n} \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & & \end{array} \middle| \begin{array}{c} D_p - D_q \end{array} \right) \text{ and } q - q' = \left(\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline q_{21} & & & \\ \vdots & & & \\ q_{m1} & & & \end{array} \middle| \begin{array}{c} 0 \end{array} \right),$$

so $\overline{pq'}$ and $\overline{qq'}$ are orthogonal. Hence we have that p, q and q' form a right triangle with \overline{pq} as hypotenuse, which implies that $d_{out}(p, p') \leq d_{out}(p, q')$ and $d_{out}(p', q') \leq d_{out}(p, q')$. When we combine this with the previous paragraph it follows that $d_{out}(p, p'), d_{out}(q', p'), d_{out}(q, q') \leq d_{out}(p, q)$, and then using this in inequality (2) the result follows. \square

We have now considered all possible pairs of p, q , and we can then combine the results to get the main theorem.

Theorem 3.4. *The model determinantal singularity $M_{m,n}^t$ is Lipschitz normally embedded, with a bilipschitz constant $2t - 2$.*

Proof. Let $p, q \in M_{m,n}^t$. If $\langle p, q \rangle = 0$ then $d_{in}(p, q) \leq 2d_{out}(p, q)$ by Lemma 3.2. If $\langle p, q \rangle \neq 0$ then $d_{in}(p, q) \leq 2 \text{rank}(p) d_{out}(p, q)$ by Lemma 3.3. Hence in all cases $d_{in}(p, q) \leq (2t - 2) d_{out}(p, q)$ since $\text{rank } p \leq t - 1$. \square

4. THE GENERAL CASE

The case of a general determinantal singularity is much more difficult than the case of a model one. One can in general not expect a determinantal singularity to be Lipschitz normally embedded, the easiest way to see this is to note that all ICIS are determinantal, and that there are many ICIS that are not Lipschitz normally embedded. For example among the simple surface singularities A_n , D_n , E_6 , E_7 and E_8 only the A_n 's are Lipschitz normally embedded. Since the structure of determinantal singularities does not give us any new tools to study ICIS, we will probably not be able to say when an ICIS is Lipschitz normally embedded. This means that since $F^{-1}(M_{m,n}^1)$ is often an ICIS, we probably have to assume it is Lipschitz normally embedded to say anything about whether $F^{-1}(M_{m,n}^t)$ is Lipschitz normally embedded. But before we discuss such assumption further, we will give some examples of determinantal singularities that fail to be Lipschitz normally embedded.

Example 4.1. Let X be the determinantal singularity of type $(3, 3, 3)$ given by the following map $F: \mathbb{C}^3 \rightarrow M_{3,3}$:

$$F(x, y, z) = \begin{pmatrix} x & 0 & z \\ y & x & 0 \\ 0 & y & x \end{pmatrix}.$$

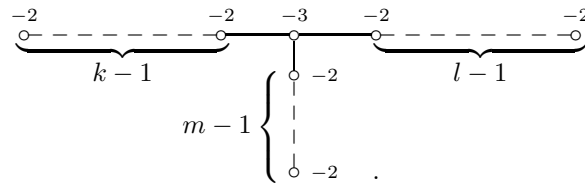
Since this is a linear embedding of \mathbb{C}^3 into \mathbb{C}^9 , one can see X as an intersection of a linear subspace and $M_{3,3}^3$. Hence one would expect it to be a nice space. On the other hand $X = V(x^3 - y^2z)$, hence it is a family of cusps degenerating to a line, or seeing another way as a cone over a cusp. But X being Lipschitz normally embedded would imply that the cusp $x^3 - y^2 = 0$ is Lipschitz normally embedded by Proposition 2.3, which we know it is not by the work of Pham and Teissier [PT69].

Notice that in the Example 4.1, $X^1 = F^{-1}(M_{3,3}^1)$ is a point and $X^1 = F^{-1}(M_{3,3}^2)$ is a line, so both X^1 and X^2 are Lipschitz normally embedded. So it does not in general follow that if X^i is Lipschitz normally embedded then X^{i+1} is. Now the singularity in Example 4.1 is not an EIDS, F is not transverse to the strata of $M_{3,3}^3$ at points on the z -axis. In the next example we will see that EIDS is not enough either.

Example 4.2 (Simple Cohen-Macaulay codimensional 2 surface singularities). In [FKN10] Frühbis-Krüger and Neumer classify simple Cohen-Macaulay codimension 2 singularities. They are all EIDS of type $(3, 2, 2)$, and the surfaces correspond to the rational triple points classified by Tjurina [Tju68]. We will look closer at two of such families. First we have the family given by the matrices:

$$\begin{pmatrix} z & y + w^l & w^m \\ w^k & y & x \end{pmatrix}.$$

This family corresponds to the family of triple points in [Tju68] called $A_{k-1, l-1, m-1}$. Tjurina shows that the dual resolution graph of their minimal resolution are:

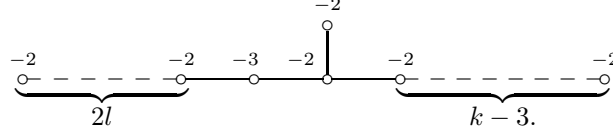


Using Remark 2.3 of [Spi90] we see that these singularities are minimal, and hence by the result of [NPP15] we get that they are Lipschitz normally embedded.

The second family is given by the matrices:

$$\begin{pmatrix} z & y + w^l & xw \\ w^k & x & y \end{pmatrix}.$$

Tjurina calls this family $B_{2l,k-1}$ and give the dual resolution graphs of their minimal resolutions as:



Following Spivakovsky this is not a minimal singularity, and since it is rational according to Tjurina it is not Lipschitz normally embedded by the result of [NPP15].

These two families do not look very different but one is Lipschitz normally embedded and the other is not. We can do the same for all the Cohen-Macaulay codimension 2 surfaces, and using the results in [NPP15] that rational surface singularities are Lipschitz normally embedded if and only if they are minimal, we get that only the family $A_{l,k,m}$ is Lipschitz normally embedded. This is similar to the case of codimension 1, since only the A_n singularities are Lipschitz normally embedded among the simple singularities.

So as we see in Example 4.2 being an EIDS with singular set Lipschitz normally embedded, is not enough to ensure the variety is Lipschitz normally embedded. One should notice that the varieties in Example 4.1 and 4.2 are both defined by maps $F: \mathbb{C}^N \rightarrow M_{m,n}$ where $N < mn$. This means that one should think of the singularity as a section of $M_{m,n}^t$, but being a subspace of a Lipschitz normally embedded space does not imply the Lipschitz normally embedded condition. If $N \geq mn$ then one can think about the singularity being a fibration over $M_{m,n}^t$, and as we saw in Proposition 2.2 products of Lipschitz normally embedded spaces are Lipschitz normally embedded. Now in this case $X^1 = F^{-1}(M_{m,n}^1)$ is ICIS if X is an EIDS, which means that we probably can not say anything general about whether it is Lipschitz normally embedded or not. So natural assumptions would be to assume that X is an EIDS and that X^1 is Lipschitz normally embedded.

ACKNOWLEDGEMENTS

The authors would like to thank Anne Pichon for first letting us know about Asuf Shachar's question on Mathoverflow.org, and later pointing out a mistake in the proof of the first version of Proposition 3.3. We would also like to thank Lev Birbrair for sending us an early version of the paper by Katz, Katz, Kerner, Liokumovich and Solomon [KKKLS16], and encouraging us to work on the problem. The first author was supported by FAPESP grant 2015/08026-4 and the second author was partially supported by FAPESP grant 2014/00304-2 and CNPq grant 306306/2015-8.

REFERENCES

- [BF08] Lev Birbrair and Alexandre Fernandes. Inner metric geometry of complex algebraic surfaces with isolated singularities. *Comm. Pure Appl. Math.*, 61(11):1483–1494, 2008.
- [BFLS16] Lev Birbrair, Alexandre Fernandes, Dũng Tráng Lê, and J. Edson Sampaio. Lipschitz regular complex algebraic sets are smooth. *Proc. Amer. Math. Soc.*, 144(3):983–987, 2016.

- [BNP14] Lev Birbrair, Walter D. Neumann, and Anne Pichon. The thick-thin decomposition and the bilipschitz classification of normal surface singularities. *Acta Math.*, 212(2):199–256, 2014.
- [DP14] James Damon and Brian Pike. Solvable groups, free divisors and nonisolated matrix singularities II: Vanishing topology. *Geom. Topol.*, 18(2):911–962, 2014.
- [Fer03] Alexandre Fernandes. Topological equivalence of complex curves and bi-Lipschitz homeomorphisms. *Michigan Math. J.*, 51(3):593–606, 2003.
- [FKN10] Anne Frühbis-Krüger and Alexander Neumer. Simple Cohen-Macaulay codimension 2 singularities. *Comm. Algebra*, 38(2):454–495, 2010.
- [FZ15] Anne Fruehbs-Krueger and Matthias Zach. On the Vanishing Topology of Isolated Cohen-Macaulay Codimension 2 Singularities. *ArXiv e-prints*, January 2015.
- [GR15] Terence Gaffney and Antoni Rangachev. Pairs of modules and determinantal isolated singularities. *ArXiv e-prints*, December 2015.
- [GZÈ09] Sabir M. Gusein-Zade and Wolfgang Èbeling. On the indices of 1-forms on determinantal singularities. *Tr. Mat. Inst. Steklova*, 267(Osobennosti i Prilozheniya):119–131, 2009.
- [KKKLS16] Karin U. Katz, Mikhail G. Katz, Dmitry Kerner, Yevgeny Liokumovich, and Jake P. Solomon. Determinantal variety and bilipschitz equivalence. *ArXiv e-prints*, February 2016.
- [MP98] Robert MacPherson and Claudio Procesi. Making conical compactifications wonderful. *Selecta Math. (N.S.)*, 4(1):125–139, 1998.
- [NBOOT13] Juan J. Nuño-Ballesteros, Bruna Oréface-Okamoto, and João N. Tomazella. The vanishing Euler characteristic of an isolated determinantal singularity. *Israel J. Math.*, 197(1):475–495, 2013.
- [NP14] Walter D. Neumann and Anne Pichon. Lipschitz geometry of complex surfaces: analytic invariants and equisingularity. *arxiv:1211.4897*, 2014.
- [NPP15] Walter D. Neumann, Helge Møller Pedersen, and Anne Pichon. Minimal surface singularities are Lipschitz normally embedded. *arxiv:1503.03301*, 2015.
- [PT69] Frédéric Pham and Bernard Teissier. Fractions Lipschitziennes d’une algèbre analytique complexe et saturation de Zariski, par Frédéric Pham et Bernard Teissier. 42 pages. Ce travail est la base de l’exposé de Frédéric Pham au Congrès International des Mathématiciens, Nice 1970., June 1969.
- [Spi90] Mark Spivakovsky. Sandwiched singularities and desingularization of surfaces by normalized Nash transformations. *Ann. of Math. (2)*, 131(3):411–491, 1990.
- [SRDSP14] Maria Aparecida Soares Ruas and Miriam Da Silva Pereira. Codimension two determinantal varieties with isolated singularities. *Math. Scand.*, 115(2):161–172, 2014.
- [Tju68] Galina N. Tjurina. Absolute isolation of rational singularities, and triple rational points. *Funkcional. Anal. i Priložen.*, 2(4):70–81, 1968.

ICMC-USP

ICMC-USP

E-mail address: helge@imf.au.dk

E-mail address: maasruas@icmc.usp.br